# Linear Complementarity Problem Simulation of The Meter Stick Trick

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*Abstract*— **In this paper, we focus on naturally incorporating contact modes estimation in a dynamical simulation of an object with multiple contacts. More specifically, this paper presents an algorithm to simulate a meter stick trick using a Linear Complementarity Problem (LCP) formulation. A meter stick, modeled as a planar uniform horizontal rod in gravity, is supported by two point contacts or "fingers". We developed a LCP-based algorithm to simulate motion of the meter stick when one finger moves toward the other at a fixed speed. The simulation matches observations of physical meter stick trick experiments. Overall, this paper shows that a LCP-based algorithm is a robust option to naturally capture the switching contact modes in a dynamical simulation of an object with multiple contacts.**

## I. Introduction

Robotic manipulation plays a significant role in many applications including the manufacturing and health care industry. Manipulation planning is a challenging task because of the complexity in understanding and modeling the relationship between a manipulator and its environment. Even when the mathematical model of the robotic system is assumed to be completely known, fully understanding and simulating the interactions between a manipulator and its environment is not trivial. One of the difficulties is to estimate contact modes between a manipulator in contact with an object. Contact mode estimation is important for manipulation planning because often time contact modes will determine the amount of force to apply or the next course of action to be taken by the robot. For example, when a robotic hand is grasping an object, the contact modes of the fingers on the object differs depending on the amount of forces applied. If the forces applied by each finger is too small, the object slides. Knowing that the contacts are in sliding mode is helpful for the robotic hand to respond accordingly to achieve a firm grasp.

In this paper, we focus on naturally incorporating contact modes estimation in a dynamical simulation of an object with multiple contacts. More specifically, this paper presents an algorithm to simulate a meter stick trick using a Linear Complementarity Problem (LCP) formulation. A meter stick, modeled as a planar uniform horizontal rod in gravity, is supported by two point contacts or "fingers". We developed a LCP-based algorithm to simulate motion of the meter stick when one finger moves toward the other at a fixed speed. The simulation shows that at slow speed, the meter stick always stays balanced between the fingers. The meter stick alternates between sticking and sliding modes until both fingers reach the center of mass of the meter stick. At high speed, the simulation shows that the meter stick falls. Both cases match observations of physical meter stick trick experiments. Overall, this paper shows that a LCP-based algorithm is a robust option to naturally capture the switching contact modes in a dynamical simulation of an object with multiple contacts.

The rest of the paper is laid out as follows: in Section II, we present a review of Berard and colleagues' work in using LCP to define contact modes; in Section III, we discuss the mathematical background of the LCP formulation and dynamic model in the plane; in Section IV, we formulate a discretized dynamic model in LCP form; in Sections V, we discuss the meter stick trick and results from the LCP-based simulation of the trick; and in Section VI, we conclude with remarks on future works.

## II. Related Works

Recently, Berard et. al. [1] used a linear complementarity problem formulation of rigid body dynamics to motivate definitions of contact modes. Extending a method to compute the set of all wrenches guaranteed to achieve a particular contact state for a three-dimensional system is difficult. In the plane, possible interactions at a contact are *slide left, slide right, roll*, and *separate*. However for a contact in a three-dimensional space, there is an infinite number of directions that a contact could slide. The authors used a linear complementarity problem formulation to obtain definitions of contact modes for a planar system. Then, they extended the definitions to a three-dimensional system. In the paper, they showed the equivalence between intuitively motivated contact modes and those implied directly by the LCP.

The paper begins by presenting the background of dynamic model of a rigid body in the plane, which includes Newton-Euler equations, kinematic equations, normal complementarity and friction law. It's a pretty detailed form of background, since if we have all the equations mentioned above, we can characterize a system mathematically so as to perform the next step calculation. General manipulation tasks can be viewed as a sequence of contact modes leading from an initial state to a goal state. So the paper also explains different contact modes in the plane. To help solidify the ideas of contact modes, the paper gives an example showing a disc initially at rest and in contact with a small fixed disc, which illustrates intuitive contact modes for an object clearly. From this part, we can better understand how the contact forces and accelerations change corresponding to different types of contact modes. Then, the paper shows a concise definition of LCP and rewrites the instantaneous dynamic model into LCP formulation for two dimensional system. In addition, the paper shows the intuitively-motivated contact modes are identical to those obtained by the LCP formulation. One contribution of the paper is that it extends LCP-motivated contact modes into three-dimensional, which is a great challenge for previous methods.

Overall, this paper presents the equality of intuitive contact modes for rigid bodies with the cones generated from a linear complementarity formulation of the dynamics. Besides, the paper shows the system as a LCP and shows how this formulation easily reduces to polyhedral convex cones by using multiple friction models.

In addition to [1], we referred to [2] which provides an iterative framework to formulate a dynamical system simulation in a mixed complementarity problem (MCP) form. Another paper [3] includes the method to rewrite the MCP into a LCP form which can then be solved using a Lemke algorithm.

## III. Background

This section explains the mathematical background related to the LCP-based simulation. The LCP formulation, dynamic model in the plane and dynamic time-stepping are discussed.

### A. LCP Formulation

The standard linear complementarity problem is defined as follows. Given the constant matrix $\boldsymbol{B} \in \mathbb{R}^{m\times m}$ and vector $\mathbf{b} \in \mathbb{R}^m$, find vectors $\boldsymbol{z} \in \mathbb{R}^m$, $\mathbf{y} \in \mathbb{R}^m$ such that

$$\begin{aligned} \boldsymbol{y} &= \boldsymbol{B}\boldsymbol{z} + \boldsymbol{b} \\ 0 \leq \boldsymbol{y} &\perp \boldsymbol{z} \geq 0 \end{aligned} \tag{1}$$

where elements of $\mathbf{y}$ and $\mathbf{z}$ are greater than 0. This constraint on $\mathbf{y}$ and $\mathbf{z}$ requires the use of slack variables.

For every pair of elements in $\mathbf{y}$ and $\mathbf{z}$ that can be both positive and negative, a pair of slack variables are defined to replace the original element.

$$\begin{aligned} y_i &= y_i^+ - y_i^- \\ z_i &= z_i^+ - z_i^- \end{aligned}$$

where $y_i^+$ is the positive component of $y_i$, and $y_i^-$ is the negative component of $y_i$. In other words, when $y_i$ is positive, $y_i^+ > 0$ and $y_i^- = 0$. Likewise, when $y_i$ is negative, $y_i^+ = 0$ and $y_i^- > 0$. Hence, a original vector of $[y_1\ y_2\ ...\ y_n]^T$ is rewritten as $[y_1\ y_2^+\ y_2^-\ ...\ y_n]^T$ if $y_2$ is allowed to have either a positive value or a negatives value. The same analysis is also applied to $z_i$.

### B. Dynamic Model In The Plane

A dynamic model is required to describe the dynamics of a system. In this work, a dynamic model in a plane is considered, and it includes both the dynamics of a body and contact constraints. The notation and formulation of the model is analogous to [1], [2].

*1) Newton-Euler Equations:* Dynamics of a body in a plane is described using the Newton-Euler equation:

$$\mathbf{M}\dot{\mathbf{v}} = \mathbf{W}_n\boldsymbol{\lambda}_n + \mathbf{W}_f\boldsymbol{\lambda}_f + \mathbf{w}_{ext} \tag{2}$$

where $\mathbf{M} = diag(m, m, J)$ is the generalized mass matrix of the body, $\mathbf{w}_{ext} = [f_x\ f_y\ \tau_z]^T$ is the external wrench applied to the body. The moment of inertia, $J$, of a slander rod is $\frac{1}{12}mL^2$. In addition, $\boldsymbol{\lambda}_n$ and $\boldsymbol{\lambda}_f$ are vectors containing all the normal and tangential components of the contact forces applied to the body. The distance between a finger and the center of mass of a body is defined as $\mathbf{r}_i$, the normal contact forces on the body is defined as $\mathbf{n}_i$, and the tangential contact forces on the body is defined as $\mathbf{t}_i$. The $x$ and $y$ components of $\mathbf{r}_i$ and $\mathbf{n}_i$ are $r_{ix}$, $r_{iy}$, $n_{ix}$, and $n_{iy}$ respectively. $\mathbf{W}_n$ and $\mathbf{W}_f$ are Jacobian matrices that map the contact forces to their equivalent wrenches in the body-fixed frame. They are defined as:

$$\begin{aligned} \mathbf{W}_n &= \begin{bmatrix} ... & \mathbf{n}_i & ... \\ ... & \mathbf{r}_i \bigotimes \mathbf{n}_i & ... \end{bmatrix} \\ \mathbf{W}_f &= \begin{bmatrix} ... & \mathbf{t}_i & ... \\ ... & \mathbf{r}_i \bigotimes \mathbf{t}_i & ... \end{bmatrix} \end{aligned} \tag{3}$$

where $\mathbf{r}_i \bigotimes \mathbf{t}_i$ is defined as $r_{ix}t_{iy} - r_{iy}t_{ix}$.

The velocity of the body in the configuration space is related to the system velocity based on the relationship:

$$\dot{\mathbf{q}} = \mathbf{G}\mathbf{v} \tag{4}$$

where $\mathbf{q}$ represents the configuration of the body and $\mathbf{G}$ is the representation Jacoban relating the system velocity $\mathbf{v}$ to the time-derivative of the system configuration $\dot{\mathbf{q}}$.

*2) Normal Contact Constraints:* For each contact, we define the distance along the normal direction between a body and a stationary contact as $\boldsymbol{\Psi}_{i,n}$ and the normal contact force as $\boldsymbol{\lambda}_{i,n}$. When the contact force is greater than zero then the normal distance equals to zero. Otherwise, when the normal distance is greater than zero then the contact force equals to zero. Hence, the constraint can be written as:

$$0 \leq \boldsymbol{\Psi}_n \perp \boldsymbol{\lambda}_n \geq 0 \tag{5}$$

where the symbol $\perp$ indicates normality (i.e., $\boldsymbol{\Psi}_n^T\boldsymbol{\lambda}_n = 0$).

*3) Tangential Contact Constraints:* For each contact, we also define the distance along the tangential direction between a body and a contact as $\boldsymbol{\Psi}_{i,f}$ and the tangential contact force as $\boldsymbol{\lambda}_{i,f}$. Combining the tangential force vectors and relative slip velocity vectors at a contact into single vectors, maximum dissipation for all contacts can be written compactly as:

$$0 \leq \boldsymbol{\lambda}_f \perp \mathbf{W}_f^T\mathbf{v} + \mathbf{E}\boldsymbol{\sigma} + \frac{\partial \boldsymbol{\Psi}_f}{\partial t} \geq 0 \tag{6}$$

$$0 \leq \boldsymbol{\sigma} \perp \mathbf{U}\boldsymbol{\lambda}_n - \mathbf{E}^T\boldsymbol{\lambda}_f \geq 0 \tag{7}$$

where $\mathbf{U}$ is the diagonal matrix with the $i^{th}$ diagonal element equal to $\mu_i$, $\sigma_i$ approximates the sliding speed at contact $i$, and $\mathbf{E}$ is the block diagonal matrix with $i^{th}$ block on the main diagonal [2].

## IV. Discretized Dynamic Model in LCP Form

To simulate the dynamics of a body with multiple contacts, we utilize the complementarity between contact forces and the relative motions between contacts and the body which is discussed in Section III. Because of the complementarity, LCP is suitable for computing the dynamics of a body with multiple contacts. Using dynamic time-stepping, we simulate a system's dynamics iteratively. The rest of this section described the dynamic time-stepping equations and the method to formulate the system's dynamics in LCP form.

### A. Dynamic Time-stepping Equations

Simulating the dynamics of a system iteratively requires discretization of the continuous dynamic model. Using the Euler's method, derivatives are approximated as:

$$\dot{\mathbf{q}} = \frac{\mathbf{q}^{l+1} - \mathbf{q}^{l}}{h} \tag{8}$$

where time step $h$ is a constant.

The Newton-Euler equation (2) and velocity kinematic equation (4) are rewritten in a discrete time form as follows:

$$\begin{aligned} \mathbf{M}\mathbf{v}^{l+1} &= \mathbf{M}\mathbf{v}^{l} + h(\boldsymbol{\lambda}_{ext}^{l+1} + \mathbf{W}_n\boldsymbol{\lambda}_n^{l+1} + \mathbf{W}_f\boldsymbol{\lambda}_f^{l+1}) \\ \mathbf{q}^{l+1} &= \mathbf{q}^{l} + h\mathbf{v}^{l+1} \end{aligned} \tag{9}$$

where $\boldsymbol{\lambda}$ represents a force wrench. All applied or non-constraint forces are included in $\boldsymbol{\lambda}_{ext}$.

The discrete forms of the non-penetration and friction constraints (5), (6), and (7) are:

$$0 \leq \boldsymbol{\lambda}_n^{l+1} \perp \boldsymbol{\Psi}_n^{l} + \frac{\partial \boldsymbol{\Psi}_n^{l}}{\partial \boldsymbol{q}}\Delta\boldsymbol{q} + \frac{\partial \boldsymbol{\Psi}_n^{l}}{\partial t}\Delta t \geq 0 \tag{10}$$

$$0 \leq \boldsymbol{\lambda}_f^{l+1} \perp \boldsymbol{E}\boldsymbol{\sigma}^{l+1} + \frac{\partial \boldsymbol{\Psi}_f^{l}}{\partial \boldsymbol{q}}\Delta\boldsymbol{q} + \frac{\partial \boldsymbol{\Psi}_f^{l}}{\partial t}\Delta t \geq 0 \tag{11}$$

$$0 \leq \boldsymbol{\sigma}^{l+1} \perp \boldsymbol{U}\boldsymbol{\lambda}_n^{l+1} - \boldsymbol{E}^T\boldsymbol{\lambda}_f^{l+1} \geq 0 \tag{12}$$

where $\Delta\boldsymbol{q} = \boldsymbol{q}^{l+1} - \boldsymbol{q}^{l}$, $\Delta t = h$, $\frac{\partial \boldsymbol{\Psi}_f^{l}}{\partial \boldsymbol{q}} = \boldsymbol{W}_f^T$, and $\frac{\partial \boldsymbol{\Psi}_n^{l}}{\partial \boldsymbol{q}} = \boldsymbol{W}_n^T$. Note that $\frac{\partial \boldsymbol{\Psi}_f^{l}}{\partial t}\Delta t$ represents the lateral position change of the frictional surface in one time step.

Rewrite (10), (11) and (12) in term of the generalized velocity vector $\boldsymbol{v}$ and the contact impulse in term of $p_{(.)} = h\lambda_{(.)}$. We arrive at the following dynamic time-stepping equations:

$$\begin{aligned} 0 \leq \boldsymbol{p}_n^{l+1} \perp \frac{\boldsymbol{\Psi}_n^{l}}{h} + \boldsymbol{W}_n^T\boldsymbol{v}^{l+1} + \frac{\partial \boldsymbol{\Psi}_n^{l}}{\partial t} \geq 0] \\ 0 \leq \boldsymbol{p}_f^{l+1} \perp \frac{\boldsymbol{E}\boldsymbol{\sigma}^{l+1}}{h} + \boldsymbol{W}_f^T\boldsymbol{v}^{l+1} + \frac{\partial \boldsymbol{\Psi}_f^{l}}{\partial t} \geq 0 \\ 0 \leq \boldsymbol{\sigma}^{l+1} \perp \boldsymbol{U}\boldsymbol{\lambda}_n^{l+1} - \boldsymbol{E}^T\boldsymbol{\lambda}_f^{l+1} \geq 0 \end{aligned} \tag{13}$$

where the equations are in complementarity forms similar to (1). They are supplemented by the discretized dynamic equations in (9) which is rewritten as:

$$\begin{aligned} \mathbf{M}\mathbf{v}^{l+1} &= \mathbf{M}\mathbf{v}^{l} + \boldsymbol{p}_{ext} + \mathbf{W}_n\boldsymbol{p}_n^{l+1} + \mathbf{W}_f\boldsymbol{p}_f^{l+1} \\ \mathbf{q}^{l+1} &= \mathbf{q}^{l} + h\mathbf{v}^{l+1} \end{aligned} \tag{14}$$

### B. Dynamics in LCP Form

Equations (13) and (14) constitute a MCP (15). In order to solve (15) to simulate the dynamics of a body with multiple contacts, we rewrite it into LCP form by substituting away the $\mathbf{v}^{l+1}$ term based on the relationship from (14). Rewriting (14) to obtain

$$\begin{aligned} \mathbf{v}^{l+1} = \mathbf{M}^{-1}\mathbf{W}_n\mathbf{p}_n^{l+1} \\ + \mathbf{M}^{-1}\mathbf{W}_f\mathbf{p}_f^{l+1} + \mathbf{v}^{l} + \mathbf{M}^{-1}\mathbf{p}_{ext} \end{aligned} \tag{16}$$

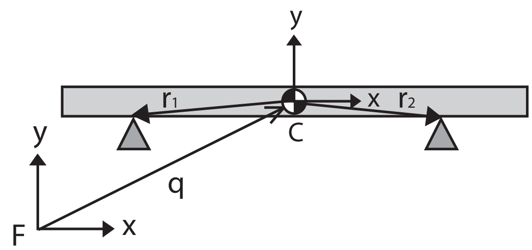


Fig. 1. Configuration of a meter stick trick. C denotes the body-fixed frame, and F denotes the world frame.

From the MCP, we have the following equations:

$$\begin{aligned} \boldsymbol{\rho}_n^{l+1} &= \mathbf{W}_n^T\mathbf{v}^{l+1} + \frac{\boldsymbol{\Psi}_n^{l}}{h} + \frac{\partial \boldsymbol{\Psi}_n^{l}}{\partial t} \\ \boldsymbol{\rho}_f^{l+1} &= \mathbf{W}_f^T\mathbf{v}^{l+1} + \frac{\partial \boldsymbol{\Psi}_f^{l}}{\partial t} \\ \mathbf{s}^{l+1} &= \mathbf{U}\mathbf{p}_n^{l+1} - \mathbf{E}^T\mathbf{p}_f^{l+1} \end{aligned} \tag{17}$$

Substituting (16) into (17) and rewriting the equations in matrix form, the MCP is converted into a LCP (18). The LCP is solved using the Lemke algorithm [4]. The resulting variable values at each iteration are used to update the variables at the next iteration.

## V. The Meter Stick Trick

LCP-based dynamics simulation described in Section IV is used to simulate a meter stick trick. This section presents the model of the meter stick trick and discusses the results from the simulations.

### A. Model of a Meter Stick Trick

A meter stick trick involves balancing a meter stick using two fingers while moving one of the finger towards the other stationary finger. At relatively low velocity, the fingers meet each other at the center of the mass of the meter stick. Fig. 1 shows the configurations of the meter stick trick system. The length of meter stick is $L = 1$ m, the thickness is $t = 0.5$ cm, and the mass is $m = 0.2$ kg. Additionally, the vector points from the original point of the world frame to the center of mass is $q$, which corresponds to the configurations of the meter stick. Hence, the relevant components in the LCP (18) for the meter stick trick are as follows:

$$\mathbf{M} = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & \frac{1}{60} \end{bmatrix} \quad \mathbf{p}_{ext} = \begin{bmatrix} 0 \\ -mg \\ 0 \end{bmatrix}$$

$$\mathbf{r}_1 = \begin{bmatrix} r_{1x} \\ -0.025 \end{bmatrix} \; \mathbf{r}_2 = \begin{bmatrix} r_{2x} \\ -0.025 \end{bmatrix}$$

$$\mathbf{W}_n = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ r_{1x} & r_{2x} \end{bmatrix}$$

$$
\begin{bmatrix} 0 \\ \boldsymbol{\rho}_n^{l+1} \\ \boldsymbol{\rho}_f^{l+1} \\ \mathbf{s}^{l+1} \end{bmatrix} = \begin{bmatrix} -\mathbf{M} & \mathbf{W}_n & \mathbf{W}_f & 0 \\ \mathbf{W}_n^T & 0 & 0 & 0 \\ \mathbf{W}_f^T & 0 & 0 & \mathbf{E} \\ 0 & \mathbf{U} & -\mathbf{E}^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{v}^{l+1} \\ \mathbf{p}_n^{l+1} \\ \mathbf{p}_f^{l+1} \\ \boldsymbol{\sigma}^{l+1} \end{bmatrix} + \begin{bmatrix} \mathbf{M}\mathbf{v}^l + \mathbf{p}_{ext} \\ \frac{\boldsymbol{\Psi}_n^l}{h} + \frac{\partial \boldsymbol{\Psi}_n^l}{\partial t} \\ \frac{\partial \boldsymbol{\Psi}_f^l}{\partial t} \\ 0 \end{bmatrix} \tag{15}
$$

$$
\begin{bmatrix} \boldsymbol{\rho}_n^{l+1} \\ \boldsymbol{\rho}_f^{l+1} \\ \boldsymbol{s}^{l+1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{W}_n^T \boldsymbol{M}^{-1} \boldsymbol{W}_n & \boldsymbol{W}_n^T \boldsymbol{M}^{-1} \boldsymbol{W}_f & 0 \\ \boldsymbol{W}_f^T \boldsymbol{M}^{-1} \boldsymbol{W}_n & \boldsymbol{W}_f^T \boldsymbol{M}^{-1} \boldsymbol{W}_f & \boldsymbol{E} \\ 0 & -\boldsymbol{E}^T & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{p}_n^{l+1} \\ \boldsymbol{p}_f^{l+1} \\ \boldsymbol{\sigma}^{l+1} \end{bmatrix} + \begin{bmatrix} \mathbf{W}_n^T \mathbf{v}^l + \mathbf{W}_n^T \mathbf{M}^{-1} \mathbf{p}_{ext} + \frac{\boldsymbol{\Psi}_n^l}{h} + \frac{\partial \boldsymbol{\Psi}_n^l}{\partial t} \\ \mathbf{W}_f^T \mathbf{v}^l + \mathbf{W}_f^T \mathbf{M}^{-1} \mathbf{p}_{ext} + \frac{\partial \boldsymbol{\Psi}_f^l}{\partial t} \\ 0 \end{bmatrix} \tag{18}
$$

$$
\mathbf{W}_f = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ -r_{1y} & r_{1y} & -r_{2y} & r_{2y} \end{bmatrix}
$$

$$
\mathbf{U} = \begin{bmatrix} \mu & 0 \\ 0 & \mu \end{bmatrix}
$$

$$
\mathbf{E} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}
$$

$$
\boldsymbol{\Psi}_n = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \frac{\partial \boldsymbol{\Psi}_f}{\partial t} = \begin{bmatrix} -v_m \\ v_m \\ 0 \\ 0 \end{bmatrix}
$$

where $\mu$ is the coefficient of friction which can be either static or kinetic depending on the contact modes of the fingers, and $v_m$ is the constant velocity of the moving finger.

In simulations, the left finger is allowed to move at a constant velocity, and the right finger stays stationary at all time. Two cases are investigated in this paper: (a) low finger velocity and (b) high finger velocity. The rest of this section discusses the results for both simulation cases.

### *B. Low Finger Velocity*

When the moving finger velocity is relatively low, the meter stick always stays balance between the two fingers. Fig. 2 and Fig. 3 show the result of a simulation of a meter stick trick whereby the two supporting fingers started off at 0.4 m away from the center of mass of the meter stick. Finger 1 is moving at a velocity of 0.1 m/s. Fig. 2 illustrates that the meter stick is alternating between sticking and sliding modes with respect to the fingers. This result is observed in a physical meter stick trick.

Note that in Fig. 3, the distance between the two fingers at the time of switching can be analytically solved for based on a simple force balance relationship. Referring to Fig. 4, the positions of both fingers satisfy the following relationship at all time:

$$
W_1 x_1 = W_2 x_2 \tag{19}
$$

where $W_i$ is the normal force applied by finger $i$ on the meter stick. When finger 1 starts to slide and finger 2 stops sliding, finger 1 has to overcome static friction to start sliding, and

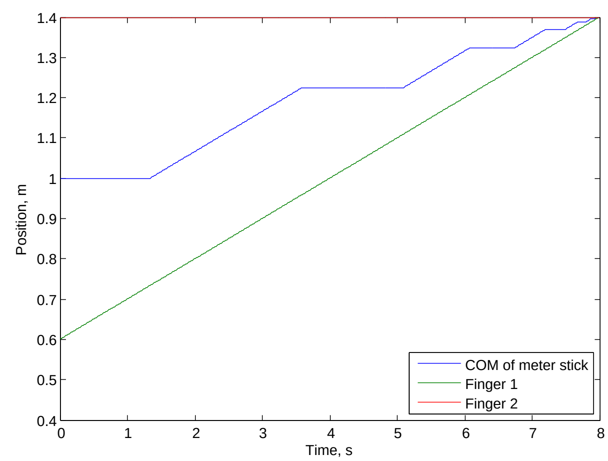


Fig. 2. Positions of the center of mass of the meter stick and the two supporting fingers.

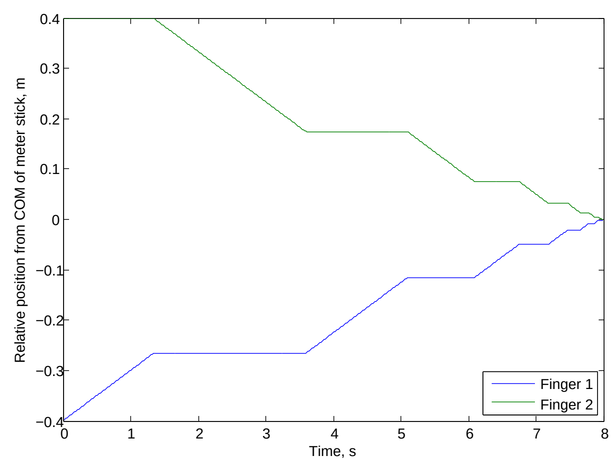


Fig. 3. Relative positions of the two supporting fingers from the center of mass of the meter stick.

finger 2 is under kinetic friction when it is sliding. Generally, kinetic friction, $\mu_k$, and static friction, $\mu_s$, are functions of the normal force, $N$, applied at the contact. This paper assumes a simple friction model whereby friction forces, $F$, follow the following equations:

$$
\begin{aligned} F_s &= \mu_s N \\ F_k &= \mu_k N. \end{aligned} \tag{20}
$$

Combining both (19) and (20), we arrive at (21) which

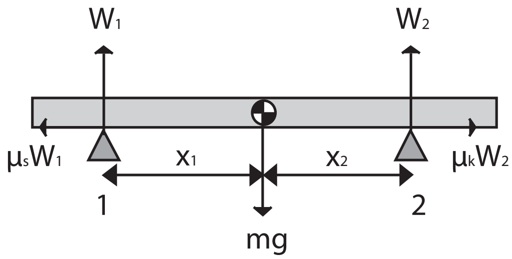


Fig. 4. Force balance on a meter stick supporting by two fingers when finger 1 is sticking to the meter stick and finger 2 is sliding relative to the meter stick.

describes the relationship between the coefficient of friction and the positions of fingers when contact mode switching occurs.

$$\mu_k x_{start} = \mu_s x_{stop} \tag{21}$$

where $x_{start}$ is the position of the finger that starts to slide, and $x_{stop}$ is the position of the finger that stops sliding.

### C. High Finger Velocity

At high finger velocity, the meter stick trick fails because the contact modes do have enough time to switch accordingly. Fig. 5 show the result of a simulation of a meter stick trick whereby the two supporting fingers started off at 0.4 m away from the center of mass of the meter stick. Velocity of finger 1 is varied. When the velocity is increased, the contact mode switching becomes less distinct, and eventually, the meter stick trick fails.

## VI. Conclusions

We have presented a dynamics simulation method that captures contact modes naturally in the dynamic time-stepping iteration using a LCP-based algorithm. The method is used to simulate a meter stick trick. A meter stick in gravity is supported by two fingers whereby one of the finger is moving at a constant velocity towards the other stationary finger. The simulation results match a physical meter stick trick. When the velocity is relatively low, the meter stick alternates between sliding mode and sticking mode until both fingers meet at the center of mass of the meter stick. When the velocity is increased, the contact mode switching becomes less distinct, and eventually, the meter stick trick fails.

Based on this work, future directions can involve extending the LCP-based dynamics simulation into a more complex system. Other than that, deriving the analogy of this simulation method in three dimensional space will open up a broader class of practical applications.

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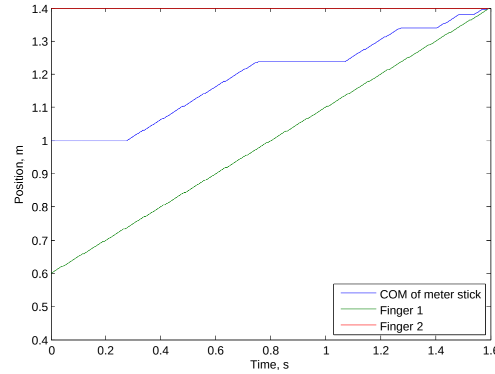
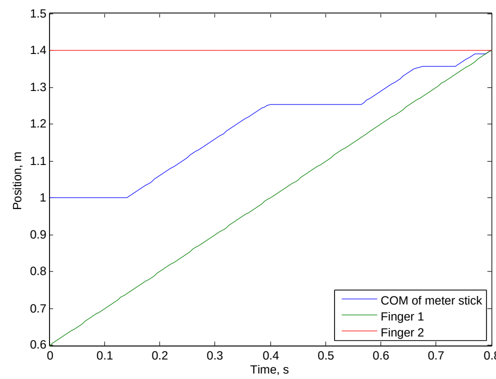
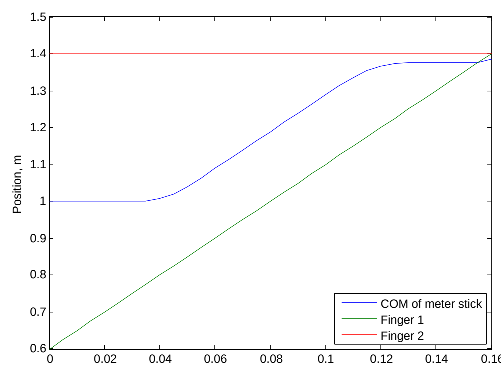


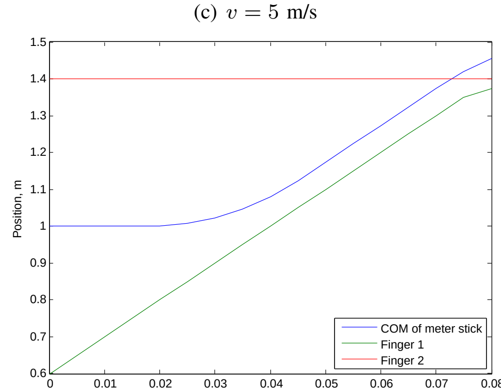

(a) $v = 0.5$ m/s




(b) $v = 1$ m/s




(c) $v = 5$ m/s




(d) $v = 10$ m/s

Fig. 5. The velocity of the moving finger is varied. When the velocity is increased, the contact mode switching becomes less distinct, and eventually, the meter stick trick fails.